

# Fixed-time consensus of multiple double-integrator systems under directed topologies: A motion-planning approach

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## Abstract

This paper investigates the fixed-time consensus problem under directed topologies. By using a motion-planning approach, a class of distributed fixed-time algorithms are developed for a multi-agent system with double-integrator dynamics. In the context of the fixed-time consensus, we focus on both directed fixed and switching topologies. Under the directed fixed topology, a novel class of distributed algorithms are designed, which guarantee the consensus of the multi-agent system with a fixed settling time if the topology has a directed spanning tree. Under the directed periodically switching topologies, the fixed-time consensus is solved via the proposed algorithms if the topologies jointly have a directed spanning tree. In particular, the fixed settling time can be off-line pre-assigned according to task requirements. Compared with the existing results, to our best knowledge, it is the first time to solve the fixed-time consensus problem for double-integrator systems under directed topologies. Finally, a numerical example is given to illustrate the effectiveness of the analytical results.

*Key words:* Distributed control; Fixed settling time; Directed spanning tree; Multi-agent system; Motion-planning approach

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## 1 Introduction

Over the past two decades, with the advent of wireless networks and powerful embedded systems, the distributed coordination of multi-agent systems has received significant attention in the control community due to its wide applications in various engineering systems such as data fusion of sensor networks, task cooperation of robots, synchronization of distributed oscillators, and formation maneuver of unmanned vehicles. As the most fundamental research topic for multi-agent coordination, consensus problems have been investigated intensively. Consensus refers to a group of agents reaching an agreement on certain quantities of interest via local interaction. By specifying desired separations among different agents, consensus algorithms can be applied to achieve distributed coordination including formation control and flocking.

The consensus problems have been primarily studied for multi-agent systems with different dynamics (see [1,2,3,4,5,6,7,8,9,10,11,12,13] and references therein). According to the rate of convergence, which is a significant performance index for evaluating the effectiveness of the designed consensus algorithms, existing consensus studies can be roughly categorized into two classes, namely, asymptotic consensus and finite-time consensus. Asymptotic consensus problems were widely investigated under different scenarios [3,4,5,6,8,14,15,16]. In [3], under directed switching topologies, asymptotic consensus problems were solved if and only if the time-varying network topologies jointly had a directed spanning tree. Recently, some conditions for second-order consensus were derived in [6,14,16]. By using adaptive control approaches, the adaptive consensus problem was studied in [4,15]. Furthermore, the consensus tracking problem of multiple Euler-Lagrange dynamics was studied in [10,17].

Different from the asymptotic consensus, achieving consensus in finite time was also studied by many researchers. The finite-time consensus problem was studied in [18] for multiple single-integrator systems, where

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the signed gradient flows of a differential function and discontinuous algorithms were used. Since then, a variety of finite-time consensus algorithms were proposed to solve the finite-time consensus problem under different scenarios (see [12,19,20,21,22,23,28,29] and references therein). In [22,23], the finite-time average consensus problem was investigated for multiple single-integrator systems. Further, a class of finite-time consensus algorithms for multiple double-integrator systems were given in [7,11,13,17,20,21]. Then, the finite-time consensus problem for multiple non-identical second-order nonlinear systems was studied in [28] with the settling time estimation. However, the settling time functions in [28] depended on initial states of the agents, which prohibited their practical applications if the knowledge of initial conditions was unavailable in advance.

Recently, the authors in [24] presented a novel class of nonlinear consensus algorithms under an undirected topology for single-integrator multi-agent networks, called fixed-time consensus which assumed uniform boundedness of a settling time regardless of the initial conditions. The results in [24] were further generalized in [26] to solve the robust fixed-time consensus problems under undirected topologies for single-integrator systems with bounded input disturbances. Due to the nonlinear nature of the fixed-time algorithms, it was very difficult to generalize the existing results for first-order systems [24,26] to multi-agent systems with more complex agent dynamics. A first attempt was made in [25] for double-integrator systems. Further, in [27], a truly distributed algorithm was given under undirected topology, which depended only on the relative measurements of the neighboring agents. Also, for multiple linear systems, the fixed-time formation problems were studied in [12] under an undirected complete graph. It is worth noting that most of the above-mentioned works were derived for multi-agent systems under undirected topologies. For the case of directed topologies, the existing algorithms in [25] depended directly on the inputs of each agent's neighbors, which led to a loop problem when there exists cycles in the graph. In practical applications, it is significant and challenging to design truly distributed fixed-time consensus algorithm based only on the relative measurements of the neighboring agents for double-integrator multi-agent systems under directed topologies.

Motivated by the above observations, by using a motion-planning approach, this paper investigates the fixed-time consensus problem of double-integrator systems under directed fixed and switching topologies, respectively. The main results of this paper extend the existing works in three aspects. Firstly, by using a motion-planning approach, a novel framework is introduced to solve the fixed-time consensus problems. In this framework, for double-integrator systems considered in this paper, compared with [24,25,26,27,12], a class of distributed algorithms are designed under a directed interaction topol-

ogy, which has a directed spanning tree. Secondly, compared with the existing results in [28], where the finite settling time can only be estimated and related to initial conditions, in this paper, with the proposed fixed-time consensus algorithms, the settling time can be off-line pre-assigned according to task requirements. Unlike the results in [24,25,26,27], the bounded settling time can be off-line designed in advance without estimations. Thirdly, the algorithms designed in this paper are based only on sampling measurements of the relative states among its neighbors, which greatly reduces the cost of the network interaction. To the best of authors' knowledge, it is the first time to solve the fixed-time consensus problems under directed fixed and switching topologies for double-integrator systems.

The remainder of this paper is organized as follows. The preliminaries are given in Section 2. The main theoretical results are established in Sections 3 and 4. A numerical example is reported in Section 5 to illustrate the theoretical results. Concluding remarks are finally given in Section 6.

## 2 Preliminaries

In this section, we introduce some preliminary knowledge of graph theory and matrix theory for the following analysis.

For a multi-agent system with  $N$  agents, a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is used to model the interaction among these agents, where  $\mathcal{V} = \{1, 2, \dots, N\}$  is the node set and  $\mathcal{E} \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$  is the edge set. An edge  $(v_i, v_j)$  is an ordered pair of vertices in  $\mathcal{V}$ , which means that agent  $j$  can receive information from agent  $i$ . If there is an edge from  $i$  to  $j$ ,  $i$  is defined as the parent node and  $j$  is defined as the child node. The neighbors of node  $i$  are denoted by  $\mathcal{N}_i = \{j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$ , and  $|\mathcal{N}_i|$  is the cardinality of  $\mathcal{N}_i$ . A directed tree is a directed graph, where every node, except for the root, has exactly one parent. A directed spanning tree of a directed graph is a directed tree formed by edges that connect all the nodes of the graph. We say that a graph has a directed spanning tree if a subset of the edges forms a directed spanning tree. The interaction topology may be dynamically changing. Therefore let  $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_s\}$  denote the set of all possible directed graphs defined for the  $N$  agents. In applications, the possible interaction topologies will likely be a subset of  $\bar{\mathcal{G}}$ . Obviously,  $\bar{\mathcal{G}}$  has finite elements. The union of a group of directed graphs  $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m\} \subset \bar{\mathcal{G}}$  is a directed graph with nodes given by  $\mathcal{V}$  and edge set given by the union of the edge sets of  $\mathcal{G}_i$ ,  $i = 1, \dots, m$ . The adjacency matrix  $A$  associated with  $\mathcal{G}$  is defined such that  $a_{ij} = 1$  if there is an edge from  $j$  to  $i$ , and  $a_{ij} = 0$  otherwise. The Laplacian matrix of the graph associated with the adjacency matrix  $A$  is

given as  $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ , where  $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}$  and

$l_{ij} = -a_{ij}$ ,  $i \neq j$ . Given a matrix  $M = [m_{ij}] \in \mathbb{R}^{N \times N}$ , it is said that  $M$  is nonnegative if all its elements  $m_{ij}$  are nonnegative, and  $M$  is positive if all its elements  $m_{ij}$  are positive. Further, if a nonnegative matrix  $M \in \mathbb{R}^{N \times N}$  satisfies  $M\mathbf{1} = \mathbf{1}$ , where  $\mathbf{1}$  represents  $[1, 1, \dots, 1]^T$  with an appropriate dimension, then it is said to be stochastic [37].

### 3 Fixed-time consensus under a directed fixed topology

In this section, the fixed-time consensus for multiple double-integrator systems is studied under a directed fixed topology.

Consider the multi-agent system with  $N$  agents labeled as  $1, 2, \dots, N$ . The dynamics of each agent is described by

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i = 1, 2, \dots, N, \quad (1)$$

where  $x_i(t) \in \mathbb{R}^n$  and  $v_i(t) \in \mathbb{R}^n$  are, respectively, the position and velocity of agent  $i$ , and  $u_i(t) \in \mathbb{R}^n$  the control input.

**Definition 1 (fixed-time consensus)** For multi-agent systems (1), the fixed-time consensus problem is said to be solved if and only if, for an off-line pre-assigned settling time  $T_s > 0$ , for any initial conditions, the positions and velocities of multi-agent systems (1) satisfy

$$\lim_{t \rightarrow T_s} \|x_i(t) - x_j(t)\| = 0, \\ \lim_{t \rightarrow T_s} \|v_i(t) - v_j(t)\| = 0, \quad \forall i, j \in \mathcal{V},$$

and  $x_i(t) = x_j(t)$ ,  $v_i(t) = v_j(t)$ , when  $t \geq T_s$ .

The main objective of this section is to design a class of distributed algorithms for multi-agent systems (1) with double-integrator dynamics such that the positions and velocities of all agents in networks reach consensus in a fixed settling time, which can be off-line pre-assigned. To achieve this objective, a motion-planning approach is used to design the following algorithm

$$u_i(t) = -\frac{6(t_{k+1} + t_k - 2t)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \\ - \frac{2(2t_{k+1} + t_k - 3t)}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)), \quad (2)$$

where  $i = 1, 2, \dots, N$ ,  $t_k \leq t < t_{k+1}$ . The time sequence is given by  $\{t_k = t_{k-1} + T_k\}$ , where  $t_0 = 0$ ,  $T_k = \frac{6}{(\pi k)^2} T_s$ ,  $k = 1, 2, \dots$ , and  $T_s$  is a finite settling time which can be off-line pre-assigned according to task requirements.

**Remark 1** It is worth mentioning that the above distributed algorithm (2) is designed based on a motion-planning approach. Concretely, consider the cost function  $J_k = \frac{1}{2} \int_{t_k}^{t_{k+1}} \sum_{i=1}^N u_i^T(t) R_i u_i(t) dt$  and the associated

$$\text{Hamiltonian function } H_k(t) = \frac{1}{2} \sum_{i=1}^N u_i^T(t) R_i u_i(t) + \sum_{i=1}^N (p_{x_i}^T(t) v_i(t) + p_{v_i}^T(t) u_i(t)), \text{ with terminal conditions}$$

$$x_i(t_{k+1}) = \frac{1}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} x_j(t_k) + x_i(t_k) \right] \\ + \frac{t_{k+1} - t_k}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} v_j(t_k) + v_i(t_k) \right], \\ v_i(t_{k+1}) = \frac{1}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} v_j(t_k) + v_i(t_k) \right], \quad i = 1, 2, \dots, N,$$

where  $p_{x_i}(t) \in \mathbb{R}^n$  and  $p_{v_i}(t) \in \mathbb{R}^n$  both represent the co-states. Solve the above optimal planning problem in light of Pontryagin's principle [38]. One obtains the consensus algorithm (2) for multi-agent systems with double-integrator dynamics (1).

**Assumption 1** Suppose that the topology  $\mathcal{G}$  among the agents is directed and has a directed spanning tree.

Before moving on, the following lemmas are firstly given.

**Lemma 1** [36] Under assumption 1, zero is a simple eigenvalue of  $\mathcal{L}$  with  $\mathbf{1}$  as an eigenvector and all of the nonzero eigenvalues are in the open right half plane.

**Lemma 2** [3] Let  $M = [m_{ij}] \in \mathbb{R}^{N \times N}$  be a stochastic matrix. If  $M$  has an eigenvalue  $\lambda = 1$  with algebraic multiplicity equal to one, and all the other eigenvalues satisfy  $|\lambda| < 1$ , then  $M$  is SIA, that is,  $\lim_{k \rightarrow \infty} M^k \rightarrow \mathbf{1} \xi^T$ , where  $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T \in \mathbb{R}^N$  satisfies  $M^T \xi = \xi$  and  $\mathbf{1}^T \xi = 1$ . Furthermore, each element of  $\xi$  is nonnegative.

Then, the following theorem provides the main result in this section.

**Theorem 1** Suppose Assumption 1 holds. For an off-line pre-assigned settling time  $T_s$ , the distributed algorithm (2) solves the fixed-time consensus problem of the multi-agent system (1) under directed fixed topologies, i.e.,  $\lim_{t \rightarrow T_s} \|x_i(t) - x_j(t)\| = 0$ ,  $\lim_{t \rightarrow T_s} \|v_i(t) - v_j(t)\| = 0$ , and  $x_i(t) = x_j(t)$ ,  $v_i(t) = v_j(t)$ , when  $t \geq T_s$ . Further, the final consensus values  $x^*(t)$  and  $v^*(t)$  are given by

$$x^*(t) = \sum_{i=1}^N \xi_i x_i(t_0) + (t - t_0) \sum_{i=1}^N \xi_i v_i(t_0),$$

$$v^*(t) = \sum_{i=1}^N \xi_i v_i(t_0), \quad (3)$$

where  $x_i(t_0)$  and  $v_i(t_0)$  are the initial states of the agents.

**Proof:** Firstly, we will prove that the states  $x_i(t)$  at time sequence  $\{t_k = t_{k-1} + T_k\}$  can achieve consensus as  $k \rightarrow \infty$ . By substituting the consensus algorithm (2) into multi-agent systems (1), the closed-loop system can be obtained as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\frac{6(t_{k+1} + t_k - 2t)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \\ &\quad - \frac{2(2t_{k+1} + t_k - 3t)}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)), \end{aligned} \quad (4)$$

where  $i = 1, 2, \dots, N$ ,  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$ . Then, by integrating (4) from  $t_k$  to  $t_{k+1}$ , one has

$$\begin{aligned} x_i(t_{k+1}) &= \frac{1}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} x_j(t_k) + x_i(t_k) \right] \\ &\quad + \frac{t_{k+1} - t_k}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} v_j(t_k) + v_i(t_k) \right], \\ v_i(t_{k+1}) &= \frac{1}{|\mathcal{N}_i| + 1} \left[ \sum_{j \in \mathcal{N}_i} v_j(t_k) + v_i(t_k) \right], \\ i &= 1, 2, \dots, N. \end{aligned} \quad (5)$$

Let  $X(t_k) = (x_1^T(t_k), x_2^T(t_k), \dots, x_N^T(t_k))^T$  and  $V(t_k) = (v_1^T(t_k), v_2^T(t_k), \dots, v_N^T(t_k))^T$ . One has

$$\begin{aligned} X(t_{k+1}) &= (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) \otimes I_n \cdot X(t_k) \\ &\quad + (t_{k+1} - t_k)(I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) \otimes I_n \cdot V(t_k), \\ V(t_{k+1}) &= (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) \otimes I_n \cdot V(t_k), \end{aligned}$$

where  $\mathcal{N} = \text{diag}(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_N)$ . Then, in the matrix form, one has

$$\begin{pmatrix} X(t_{k+1}) \\ V(t_{k+1}) \end{pmatrix} = H \otimes I_n \begin{pmatrix} X(t_k) \\ V(t_k) \end{pmatrix} = H^{k+1} \otimes I_n \begin{pmatrix} X(t_0) \\ V(t_0) \end{pmatrix},$$

where

$$H = \begin{pmatrix} (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) & (t_{k+1} - t_k)(I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) \\ 0 & (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}) \end{pmatrix},$$

and

$$H^k = \begin{pmatrix} (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L})^k & (t_k - t_0)(I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L})^k \\ 0 & (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L})^k \end{pmatrix}.$$

Under Assumption 1, the directed fixed topology  $\mathcal{G}$  has a spanning tree. Thus,  $I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}$  is a stochastic matrix. According to Lemma 1, one gets that  $I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}$  has an eigenvalue  $\lambda_1 = 1$  with algebraic multiplicity equal to one, and all the other eigenvalues satisfy  $|\lambda_i| < 1$ ,  $i = 2, \dots, N$ . Thus, it is followed from Lemma 2 that for matrix  $I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L}$ , there exists a column vector  $\xi$  such that

$$\lim_{k \rightarrow \infty} (I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L})^k = \mathbf{1}\xi^T. \quad (6)$$

Besides, according to  $\{t_k = t_{k-1} + T_k\}$  and  $T_k = \frac{6}{(\pi k)^2} T_s$ ,  $k = 1, 2, \dots$ , one has  $\lim_{k \rightarrow \infty} t_k = T_s$ . Thus,  $t_k - t_0$  is bounded. It follows that

$$\lim_{k \rightarrow \infty} (t_k - t_0)[(I_N - (\mathcal{N} + I_N)^{-1}\mathcal{L})^k - \mathbf{1}\xi^T] = 0. \quad (7)$$

Denote  $X^*(t) = \mathbf{1} \otimes x^*(t)$  and  $V^*(t) = \mathbf{1} \otimes v^*(t)$ . From (3), one has

$$\begin{pmatrix} X^*(t) \\ V^*(t) \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \mathbf{1}\xi^T & (t - t_0)\mathbf{1}\xi^T \\ 0 & \mathbf{1}\xi^T \end{pmatrix} \otimes I_n \end{bmatrix} \begin{pmatrix} X(t_0) \\ V(t_0) \end{pmatrix}.$$

It follows that

$$\begin{aligned} &\lim_{k \rightarrow \infty} \left[ \begin{pmatrix} X(t_{k+1}) \\ V(t_{k+1}) \end{pmatrix} - \begin{pmatrix} X^*(t_{k+1}) \\ V^*(t_{k+1}) \end{pmatrix} \right] \\ &= \lim_{k \rightarrow \infty} \left\{ \left[ H^{k+1} - \begin{pmatrix} \mathbf{1}\xi^T & (t_{k+1} - t_0)\mathbf{1}\xi^T \\ 0 & \mathbf{1}\xi^T \end{pmatrix} \right] \otimes I_n \right\} \\ &\quad \begin{pmatrix} X(t_0) \\ V(t_0) \end{pmatrix}. \end{aligned}$$

Thus, according to (6) and (7), one has

$$\lim_{k \rightarrow \infty} \left[ \begin{pmatrix} X(t_{k+1}) \\ V(t_{k+1}) \end{pmatrix} - \begin{pmatrix} X^*(t_{k+1}) \\ V^*(t_{k+1}) \end{pmatrix} \right] = 0.$$

Thus, one has the discrete states  $x_i(t_k)$  will achieve consensus with an exponential rate as  $k \rightarrow \infty$ , i.e.,  $\lim_{k \rightarrow \infty} \|x_i(t_k) - x_j(t_k)\| = 0$ ,  $\lim_{k \rightarrow \infty} \|v_i(t_k) - v_j(t_k)\| = 0$ ,  $i, j = 1, 2, \dots, N$ .

Secondly, for the off-line pre-assigned settling time  $T_s$ , we will prove that the discrete states  $x_i(t_k)$  can achieve fixed-time consensus as  $t_k \rightarrow T_s$ . Since  $\lim_{k \rightarrow \infty} t_k = T_s$ , one has

$$\begin{aligned} &\lim_{t_k \rightarrow T_s} \|x_i(t_k) - x_j(t_k)\| \\ &= \lim_{k \rightarrow \infty} \|x_i(t_k) - x_j(t_k)\| \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} & \lim_{t_k \rightarrow T_s} \|v_i(t_k) - v_j(t_k)\| \\ &= \lim_{k \rightarrow \infty} \|v_i(t_k) - v_j(t_k)\| \\ &= 0, \end{aligned}$$

where  $i, j = 1, 2, \dots, N$ . Therefore, for the off-line pre-assigned settling time  $T_s$ , the discrete states  $x_i(t_k)$  will achieve fixed-time consensus in exponential rate as  $t_k \rightarrow T_s$ .

Finally, we will prove that the continuous states  $x_i(t)$  can achieve fixed-time consensus as  $t \rightarrow T_s$ . By integrating equation (4) from  $t_k$  to  $t$ , it is obtained that

$$\begin{aligned} v_i(t) &= \frac{6(t_{k+1} - t)(t_k - t)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \\ &+ \frac{(4t_{k+1} - t_k - 3t)(t_k - t)}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)) \\ &+ v_i(t_k), \quad t_k \leq t < t_{k+1}. \end{aligned} \quad (8)$$

Thus,

$$\begin{aligned} & \|v_i(t) - v_j(t)\| \\ &\leq \frac{6(t_{k+1} - t)(t - t_k)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (x_i(t_k) - x_s(t_k)) \right\| \\ &+ \frac{(4t_{k+1} - t_k - 3t)(t - t_k)}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (v_i(t_k) - v_s(t_k)) \right\| \\ &+ \frac{6(t_{k+1} - t)(t - t_k)}{(t_{k+1} - t_k)^3(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (x_j(t_k) - x_s(t_k)) \right\| \\ &+ \frac{(4t_{k+1} - t_k - 3t)(t - t_k)}{(t_{k+1} - t_k)^2(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (v_j(t_k) - v_s(t_k)) \right\| \\ &+ \|v_i(t_k) - v_j(t_k)\| \\ &\leq \frac{6}{(t_{k+1} - t_k)(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (x_i(t_k) - x_s(t_k)) \right\| \\ &+ \frac{4}{(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (v_i(t_k) - v_s(t_k)) \right\| \\ &+ \frac{6}{(t_{k+1} - t_k)(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (x_j(t_k) - x_s(t_k)) \right\| \\ &+ \frac{4}{(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (v_j(t_k) - v_s(t_k)) \right\| \\ &+ \|v_i(t_k) - v_j(t_k)\|, \end{aligned}$$

where  $t_k \leq t < t_{k+1}$ . Note that  $\lim_{t_k \rightarrow T_s} \|x_i(t_k) - x_j(t_k)\| = 0$  with an exponential rate and  $\lim_{t_k \rightarrow T_s} (t_k - t_{k-1}) = \lim_{k \rightarrow \infty} \frac{6}{(\pi k)^2} T_s = 0$  with a polynomial rate.

Thus, one has

$$\lim_{t_k \rightarrow T_s} \frac{6}{(t_{k+1} - t_k)(|\mathcal{N}_i| + 1)} \left\| \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \right\| = 0.$$

Besides, according to  $\lim_{t_k \rightarrow T_s} \left\| \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \right\| = 0$  and  $\lim_{t_k \rightarrow T_s} \left\| \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)) \right\| = 0$ , one has  $\lim_{t_k \rightarrow T_s} \|v_i(t) - v_j(t)\| = 0$ . Further, integrating equation (8) from  $t_k$  to  $t$ , one gets

$$\begin{aligned} x_i(t) &= -\frac{(t - t_k)^2(3t_{k+1} - t_k - 2t)}{(t_{k+1} - t_k)^3(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \\ &+ \frac{(t - t_k)^2(t + t_k - 2t_{k+1})}{(t_{k+1} - t_k)^2(|\mathcal{N}_i| + 1)} \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)) \\ &+ v_i(t_k)(t - t_k) + x_i(t_k). \end{aligned}$$

Therefore,

$$\begin{aligned} & x_i(t) - x_j(t) \\ &= -\frac{(t - t_k)^2(3t_{k+1} - t_k - 2t)}{(t_{k+1} - t_k)^3} \\ &\cdot \left[ \frac{1}{(|\mathcal{N}_i| + 1)} \sum_{s \in \mathcal{N}_i} (x_i(t_k) - x_s(t_k)) \right. \\ &\quad \left. - \frac{1}{(|\mathcal{N}_j| + 1)} \sum_{s \in \mathcal{N}_j} (x_j(t_k) - x_s(t_k)) \right] \\ &+ \frac{(t - t_k)^2(t + t_k - 2t_{k+1})}{(t_{k+1} - t_k)^2} \\ &\cdot \left[ \frac{1}{(|\mathcal{N}_i| + 1)} \sum_{s \in \mathcal{N}_i} (v_i(t_k) - v_s(t_k)) \right. \\ &\quad \left. - \frac{1}{(|\mathcal{N}_j| + 1)} \sum_{s \in \mathcal{N}_j} (v_j(t_k) - v_s(t_k)) \right] \\ &+ (t - t_k)(v_i(t_k) - v_j(t_k)) + (x_i(t_k) - x_j(t_k)), \\ &t_k \leq t < t_{k+1}. \end{aligned}$$

Thus, one obtains

$$\begin{aligned} & \|x_i(t) - x_j(t)\| \\ &\leq 3 \left( \frac{1}{(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (x_i(t_k) - x_s(t_k)) \right\| \right. \\ &\quad \left. + \frac{1}{(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (x_j(t_k) - x_s(t_k)) \right\| \right) \\ &+ 2(t_{k+1} - t_k) \left( \frac{1}{(|\mathcal{N}_i| + 1)} \left\| \sum_{s \in \mathcal{N}_i} (v_i(t_k) - v_s(t_k)) \right\| \right. \\ &\quad \left. - \frac{1}{(|\mathcal{N}_j| + 1)} \left\| \sum_{s \in \mathcal{N}_j} (v_j(t_k) - v_s(t_k)) \right\| \right) \end{aligned}$$



$$\begin{aligned}
& + (t_{k+1} - t_k) \| (v_i(t_k) - v_j(t_k)) \| \\
& + \| (x_i(t_k) - x_j(t_k)) \|, \quad t_k \leq t < t_{k+1}.
\end{aligned}$$

Note that  $t_{k+1} - t_k \leq T_s < \infty$  is upper bounded, and

$$\lim_{t_k \rightarrow T_s} \left\| \sum_{j \in \mathcal{N}_i} (x_i(t_k) - x_j(t_k)) \right\| = 0,$$

$$\lim_{t_k \rightarrow T_s} \left\| \sum_{j \in \mathcal{N}_i} (v_i(t_k) - v_j(t_k)) \right\| = 0.$$

One has  $\lim_{t_k \rightarrow T_s} \|x_i(t) - x_j(t)\| = 0$ . Based on the above analysis, it follows that with the algorithm (2), the multi-agent systems of double-integrator dynamics (1) can achieve fixed-time consensus. The proof is completed.

**Remark 2** Under directed topologies, the finite-time consensus problem of single-integrator multi-agent systems has been solved in [30,31,32]. However, the algorithms in [30,31,32] are difficult to develop for solving the finite-time consensus problem of double-integrator multi-agent systems under directed topologies. Also, in [24,26], a fixed-time consensus algorithm is developed for integrator-type multi-agent systems under undirected topologies. In this paper, by using a motion-planning approach, a novel class of distributed algorithms are proposed to solve the finite-time or fixed-time consensus problem of double-integrator multi-agent systems under directed topologies.

**Remark 3** Compared with the existing works [21,19,11,28] on finite-time consensus problems and [27,24,26,25] on fixed-time consensus problems, in this paper, the settling time can be off-line pre-assigned according to task requirements, which not only realizes the consensus in the state space but also accurately controls the settling time in the time axis.

#### 4 Fixed-time consensus under directed periodically switching topologies

In some cases, the interaction among agents exhibits periodic phenomena, which implies that the topology among agents is periodically time-varying. Thus, we will investigate the fixed-time consensus problems of double-integrator multi-agent systems under directed periodical switching topologies. Before moving on, the following assumption is given.

**Assumption 2** For a time series  $\{t_k\}$  with  $t_0 = 0$ , there exists a corresponding directed topologies set  $\bar{\mathcal{G}} = \{\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_{m-1}\}$ . The topology among agents is periodically time-varying with the period  $m$ , (i.e.  $\mathcal{G}_{k+m} = \mathcal{G}_k$ ,  $k = 0, 1, \dots$ , and the topologies only exist at the time instant) such that across each time interval  $[t_k, t_{k+m-1})$ ,

the union of the directed interaction graphs at discrete times  $\{t_k, t_{k+1}, \dots, t_{k+m-1}\}$  has a spanning tree.

**Lemma 3** [3] If Assumption 2 holds, then there exists a column vector  $\xi$  such that

$$\prod_{k=0}^{\infty} \prod_{s=0}^{m-1} [I_N - (\mathcal{N}(km + s) + I_N)^{-1} \mathcal{L}(km + s)] = \mathbf{1} \xi^T.$$

Based on Lemma 3, we will analyze the control algorithm (2) under directed periodical switching topologies satisfying Assumption 2. In this case, the notation  $\mathcal{N}_i$  in (2) is replaced by  $\mathcal{N}_i(k)$ .

**Theorem 2** Suppose Assumption 2 holds. For an off-line pre-assigned settling time  $T_s$ , the distributed algorithm (2) solves the fixed-time consensus problem of multi-agent system (1) under directed periodical switching topologies.

**Proof:** Note that if we prove the discrete states  $x_i(t_k)$  will achieve consensus as  $k \rightarrow \infty$  with an exponential rate, then it follows from Theorem 1 that the conclusion in this theorem can be obtained. Thus, we will prove that discrete states  $x_i(t_k)$  will achieve consensus as  $k \rightarrow \infty$  with an exponential rate.

Denote  $\Pi_k = \prod_{s=0}^{m-1} [I_N - (\mathcal{N}(km + s) + I_N)^{-1} \mathcal{L}(km + s)]$ . For the above defined directed periodical switching topologies satisfying Assumption 2, one has  $\Pi_0 = \Pi_1 = \dots = \Pi_k$ . Therefore, according to Lemma 3, one has

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \Pi_0^k \\
& = \prod_{k=0}^{\infty} \prod_{s=0}^{m-1} [I_N - (\mathcal{N}(km + s) + I_N)^{-1} \mathcal{L}(km + s)] \\
& = \mathbf{1} \xi^T.
\end{aligned}$$

It follows that  $\Pi_0^k$  will convergent to  $\mathbf{1} \xi^T$  with an exponential rate as  $k \rightarrow \infty$ . Thus, for multi-agent systems (1), it follows from the proof of Theorem 1 that

$$\begin{aligned}
& \begin{pmatrix} X(t_{(k+1)m}) \\ V(t_{(k+1)m}) \end{pmatrix} \\
& = \begin{pmatrix} \Pi_k \otimes I_n & (t_{(k+1)m} - t_{km}) \Pi_k \otimes I_n \\ 0 & \Pi_k \otimes I_n \end{pmatrix} \begin{pmatrix} X(t_{km}) \\ V(t_{km}) \end{pmatrix} \\
& = \begin{pmatrix} \Pi_0^{k+1} \otimes I_n & (t_{(k+1)m} - t_0) \Pi_0^{k+1} \otimes I_n \\ 0 & \Pi_0^{k+1} \otimes I_n \end{pmatrix} \begin{pmatrix} X(t_0) \\ V(t_0) \end{pmatrix}.
\end{aligned}$$

Note that

$$\begin{pmatrix} X^*(t_{(k+1)m}) \\ V^*(t_{(k+1)m}) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1}\xi^T & (t_{(k+1)m} - t_0)\mathbf{1}\xi^T \\ 0 & \mathbf{1}\xi^T \end{pmatrix} \otimes I_n \cdot \begin{pmatrix} X(t_0) \\ V(t_0) \end{pmatrix}.$$

Similar to the proof of Theorem 1, one has

$$\lim_{k \rightarrow \infty} \left[ \begin{pmatrix} X(t_{(k+1)m}) \\ V(t_{(k+1)m}) \end{pmatrix} - \begin{pmatrix} X^*(t_{(k+1)m}) \\ V^*(t_{(k+1)m}) \end{pmatrix} \right] = 0. \quad (9)$$

Thus, states  $x_i(t_{km})$  and  $v_i(t_{km})$ ,  $i = 1, \dots, N$ , can achieve consensus as  $k \rightarrow \infty$ , respectively. Besides, note that

$$\begin{pmatrix} X(t_{km+1}) \\ V(t_{km+1}) \end{pmatrix} = \begin{pmatrix} 1 & (t_{km+1} - t_{km}) \\ 0 & 1 \end{pmatrix} \otimes [I_N - (\mathcal{N}(km) + I_N)^{-1} \mathcal{L}(km)] \otimes I_n \cdot \begin{pmatrix} X(t_{km}) \\ V(t_{km}) \end{pmatrix}.$$

Thus, one has

$$\begin{aligned} & \begin{pmatrix} X(t_{km+1}) \\ V(t_{km+1}) \end{pmatrix} - \begin{pmatrix} X^*(t_{km+1}) \\ V^*(t_{km+1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & (t_{km+1} - t_{km}) \\ 0 & 1 \end{pmatrix} \otimes [I_N - (\mathcal{N}(km) + I_N)^{-1} \mathcal{L}(km)] \otimes I_n \cdot \left[ \begin{pmatrix} X(t_{km}) \\ V(t_{km}) \end{pmatrix} - \begin{pmatrix} X^*(t_{km}) \\ V^*(t_{km}) \end{pmatrix} \right]. \end{aligned}$$

Since the boundness of  $t_{km+1} - t_{km}$ , it follows from (9) that

$$\lim_{k \rightarrow \infty} \left[ \begin{pmatrix} X(t_{km+1}) \\ V(t_{km+1}) \end{pmatrix} - \begin{pmatrix} X^*(t_{km+1}) \\ V^*(t_{km+1}) \end{pmatrix} \right] = 0.$$

Similarly, one has

$$\lim_{k \rightarrow \infty} \left[ \begin{pmatrix} X(t_{km+s}) \\ V(t_{km+s}) \end{pmatrix} - \begin{pmatrix} X^*(t_{km+s}) \\ V^*(t_{km+s}) \end{pmatrix} \right] = 0,$$

where  $s = 1, 2, \dots, m-1$ . Thus, one has  $\lim_{k \rightarrow \infty} \|x_i(t_k) - x_j(t_k)\| = 0$ ,  $\lim_{k \rightarrow \infty} \|v_i(t_k) - v_j(t_k)\| = 0$ ,  $i, j = 1, 2, \dots, N$ , with an exponential rate. By the derivations similar to Theorem 1, it is easy to obtain that  $\lim_{t \rightarrow T_s} \|x_i(t) - x_j(t)\| = 0$ ,  $\lim_{t \rightarrow T_s} \|v_i(t) - v_j(t)\| = 0$ ,  $i, j = 1, \dots, N$ . The proof is completed.

**Remark 4** Note that the finite-time and fixed-time consensus problems were investigated in some interesting papers [7, 19, 20, 21, 13, 25, 12, 28, 29]. To the best of the authors' knowledge, under directed topologies, it is the first

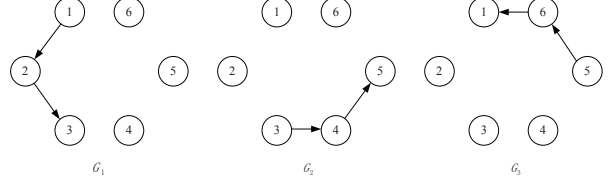


Fig. 1. The directed periodical switching topologies among 6 agents described by (1).

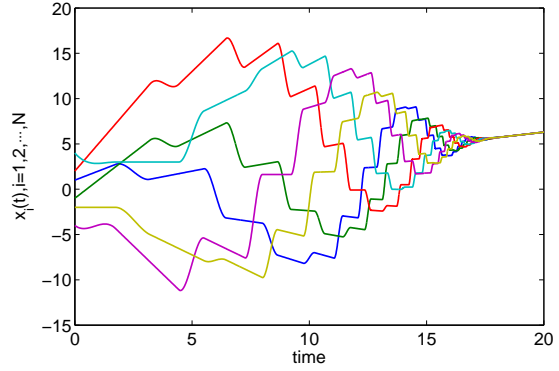


Fig. 2. The positions of multi-agent system (1) with fixed-time consensus algorithm (2) and  $T_k = \frac{6}{(\pi k)^2} T_s$ , under directed periodical switching topologies in Fig. 1.

time to solve fixed-time consensus problems for multi-agent systems with double-integrator dynamics. Besides, the fixed-time algorithms designed in this paper are based only on sampling measurements of the relative states among its neighbors, which greatly reduces the cost of the network interaction [33, 34, 35].

## 5 Simulations

In this section, an example is given to verify the theoretical results in this paper.

Consider a multi-agent system with 6 double-integrator systems described by (1). The initial states are given by  $x_1(0) = 1, x_2(0) = -1, x_3(0) = 2, x_4(0) = 4, x_5(0) = -4, x_6(0) = -2, v_1(0) = 1, v_2(0) = 2, v_3(0) = 3, v_4(0) = -3, v_5(0) = -2, v_6(0) = 0$ . For an off-line pre-assigned fixed settling time  $T_s = 20$ , the directed periodical switching topologies are shown in Fig. 1. The simulation results are given in Fig. 2-3, where the positions and velocities of multi-agent system (1) under the algorithm (2) achieve fixed-time consensus.

## 6 Conclusion

In this paper, the fixed-time consensus problem under directed topologies has been investigated for a group of agents with double-integrator dynamics. By using a motion-planning approach, a class of distributed algorithms have been constructed in this paper to solve

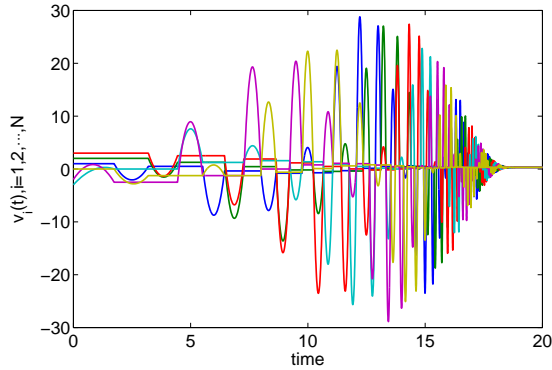


Fig. 3. The velocities of multi-agent system (1) with fixed-time consensus algorithm (2) and  $T_k = \frac{6}{(\pi k)^2} T_s$ , under directed periodical switching topologies in Fig. 1.

finite-time and fixed-time consensus problems under both the directed fixed and periodical switching topologies, respectively. Specially, the fixed settling time can be off-line pre-assigned. Future works will focus on solving distributed consensus problem for multiple agents modeled by general linear or nonlinear dynamics under directed topologies.

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